ON THE LTI PROPERTIES OF ADAPTIVE FEEDFORWARD SYSTEMS WITH TAP DELAY-LINE REGRESSORS

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Abstract

It is shown that an adaptive system whose regressor is formed by tap delay-line (TDL) filtering of a multitone sinusoidal signal is representable as a parallel connection of a linear time-invariant (LTI) block and a linear time-varying (LTV) block. A norm-bound (induced 2-norm) is computed explicitly on the LTV block and is shown to decrease as N^{-1} where N is the number of taps. Hence, the adaptive system becomes LTI in the limit as the number of taps goes to infinity. In the more realistic case where the number of taps N is finite, the new "LTI plus norm-bounded perturbation" representation renders, for the first time, the adaptive system analyzable by standard robust control methods.

adaptive control
noise cancelling

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1 INTRODUCTION

A fundamental problem in adaptive noise cancellation is that of cancelling a sinusoidal disturbance having unknown frequency content. It is known that the adaptive LMS (least-mean-square) noise cancellation algorithm can be applied to this problem when its regressor is constructed by a tap delay-line (TDL) driven by a measured multitone sinusoidal signal containing the same harmonic content as the signal to be cancelled [1]. An important analysis by Glover [2] showed that in the limit as the number of taps goes to infinity, the resulting adaptive controller can be approximated as a linear time-invariant (LTI) controller having large gains at the disturbance frequencies. Glover's LTI analysis helps to explain the noise rejection properties of adaptive LMS filters, and provides estimates of their transients, the depth of their frequency notches, their closed-loop pole locations, etc. [2].

Rigorously speaking, Glover's LTI analysis is only applicable in the limiting case of an infinite number of taps $N = \infty$. Realistically, this represents an idealization which is never satisfied in practice. To help fill in the gap, the present paper provides a precise "LTI plus norm-bounded perturbation" characterization of the adaptive system in the case of a finite number of taps. This representation clearly demonstrates the nature of convergence to an LTI system as the number of taps is increased. Furthermore, the adaptive system becomes analyzable by standard robust control methods in the more realistic and practical case where the number of taps N is finite.

Background is given in Section 2 summarizing recent results for adaptive feedforward systems with sinusoidal regressors. In Section 3, these results are specialized to adaptive systems whose regressors are formed by TDL filtering. This leads directly to a representation as a parallel connection of a linear time-invariant (LTI) block and a linear time-varying (LTV) block. A key result of the paper is an explicit norm-bound (induced 2-norm) on the LTV block (cf., Theorem 3.1). Glover's asymptotic result [2] is recovered by noting that this norm bound decreases as N^{-1} where N is the number of taps. For N finite, the norm bound is expressed as a function of the number of taps, the adaptation gain, the number of tones, and the tone spacing. Implications of this norm bound are discussed for analyzing adaptive systems in the light of modern robust control theory.

All results in this paper are taken from a recent JPL internal document [3], and have appeared in abridged form as a conference paper [4].

2 BACKGROUND

2.1 Adaptive Systems with Harmonic Regressors

The configuration to be studied is shown in Figure 2.1. An estimate $\hat{y}$ of some signal y is to be constructed as a linear combination of the elements of a regressor vector $x(t) \in R^N$, i.e.,

Estimated Signal

$$\hat{y} = w(t)^T x(t) \tag{2.1}$$

where $w(t) \in R^N$ is a parameter vector which is tuned in real-time using the adaptation algorithm,

Adaptation Algorithm

$$w = \mu\Gamma(p)[\tilde{x}(t)e(t)] \tag{2.2}$$

Here, the notation $\Gamma(p)[\cdot]$ is used to denote the multivariable transfer function $\Gamma(s) \cdot I$ where $\Gamma(s)$ is any scalar LTI transfer function in the Laplace s operator (the differential operator p will replace the Laplace operator s in all time-domain filtering expressions); the term $e(t) \in R^1$ is an error signal; $\mu > 0$ is an adaptation gain; and the signal $\tilde{x}$ is obtained by filtering the regressor x through any stable filter $F(p)$, i.e.,

Regressor Filtering

$$\tilde{x} = F(p)[x] \tag{2.3}$$

The notation $F(p)[\cdot]$ denotes the multivariable transfer function $F(s) \cdot I$ with stable scalar filter $F(s)$.

For the purposes of this paper, it will be assumed that the regressor x can be written as a linear combination of m distinct sinusoidal components $\{\omega_i\}_{i=1}^m$, $0 < \omega_1 < \omega_2 < ... < \omega_m$. Equivalently, it is assumed that there exists a matrix $\mathcal{X} \in R^{N \times 2m}$ such that,

Harmonic Regressor

$$x = \mathcal{X}c(t) \tag{2.4}$$

$$c(t) = [\sin(\omega_1 t), \cos(\omega_1 t), ..., \sin(\omega_m t), \cos(\omega_m t)]^T \in R^{2m} \tag{2.5}$$

The following definition will be useful.

DEFINITION 2.1 *The matrix $\mathcal{X}^T\mathcal{X}$ is defined as the* **confluence matrix** *associated with the harmonic regressor x in (2.4).* ∎

The name "confluence matrix" has been chosen to reflect the fact that for overparametrized regressors $x \in R^N$, $N > 2m$, the N signal channels of the regressor are effectively combined into a smaller number of $2m$ channels by properties of this matrix. The confluence matrix should not be confused with the autocorrelation matrix which is instead related to the "outer product" $\mathcal{X}\mathcal{X}^T$.

Equations (2.1)-(2.5) taken together will be referred to as a *harmonic adaptive system.* Collectively, these equations define an important open-loop mapping from the error signal e to the estimated output $\hat{y}$. Because of its importance, this mapping will be denoted by the special character $\mathcal{H}$, i.e.,

$$\hat{y} = \mathcal{H}[e] \tag{2.6}$$

The special structure of $\mathcal{H}$ is depicted in Figure 2.1.

REMARK 2.1 The definition of $\Gamma(s)$ is left intentionally general to include analysis of the gradient algorithm (i.e., with the choice $\Gamma(s) = 1/s$), the gradient algorithm with leakage (i.e., $\Gamma(s) = 1/(s+\sigma)$; $\sigma \geq 0$), proportional-plus-integral adaptation (i.e., $\Gamma(s) = k_p + k_i/s$), or arbitrary linear adaptation algorithms of the designer's choosing. Adaptation laws which are nonlinear or normalized (e.g., divided by the norm of the regressor), are not considered here since they do not have an equivalent LTI representation $\Gamma(p)$. ∎

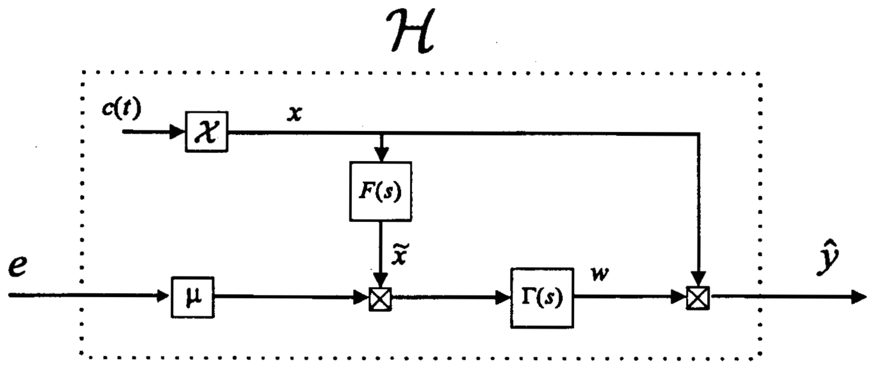

Figure 2.1: LTV operator $\hat{y} = \mathcal{H}[e]$ for adaptive system with harmonic regressor x, adaptation law $\Gamma(s)$, and regressor filter $F(s)$

REMARK 2.2 The use of the regressor filter $F(s)$ is (2.3) allows the unified treatment of many important adaptation algorithms including the well-known Filtered-X algorithm from the signal processing literature [5][6][7][8], and the Augmented Error algorithm of Monopoli [9]. Since x is comprised purely of sinusoidal components and F in (2.3) is stable, all subsequent analysis will assume that the filter output $\tilde{x}$ has reached a steady-state condition. ∎

The following result taken from [3][10] will be needed which gives necessary and sufficient conditions for the operator $\mathcal{H}$ to be LTI.

THEOREM 2.1 (LTI Representation Theorem) *Let the regressor $x(t)$ in the adaptive system (2.1)-(2.3) be given by the general multitone harmonic expression (2.4)(2.5) where the frequencies $\{\omega_i\}_{i=1}^{m}$ are distinct, nonzero, and $|F(j\omega_i)| > 0$ for all i.*

Then,

(i) *The mapping $\mathcal{H}$ from e to $\hat{y}$ is exactly representable as the linear time-invariant operator,*

$$\mathcal{H}: \quad \hat{y} = \overline{H}(p)e \tag{2.7}$$

if and only if the matrix $\mathcal{X}$ in (2.4) satisfies the following X-Orthogonality (XO) condition,

$\mathcal{X}$-Orthogonality (XO) Condition:

$$\mathcal{X}^T\mathcal{X} = D^2 \tag{2.8}$$

$$D^2 \triangleq \begin{bmatrix} {d_1}^2 \cdot I_{2\times 2} & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & {d_m}^2 \cdot I_{2\times 2} \end{bmatrix} \in R^{2m\times 2m} \tag{2.9}$$

where, ${d_i}^2 \geq 0, i = 1, \dots, m$ are scalars and $I_{2\times 2} \in R^{2\times 2}$ is the matrix identity.

(ii) *$\overline{H}(s)$ in (2.7) is given in closed-form as,*

$$\overline{H}(s) = \mu \sum_{i=1}^{m} {d_i}^2 \cdot H_i(s) \tag{2.10}$$

$$H_i(s) = \frac{F_R(i)}{2}\Big(\Gamma(s - j\omega_i) + \Gamma(s + j\omega_i)\Big) + \frac{F_I(i)}{2j}\Big(\Gamma(s - j\omega_i) - \Gamma(s + j\omega_i)\Big) \tag{2.11}$$

$$F_R(i) \triangleq \mathbf{Re}(F(j\omega_i)); \quad F_I(i) \triangleq \mathbf{Im}(F(j\omega_i)) \tag{2.12}$$

■

In words, Theorem 2.1 says that a harmonic adaptive system is LTI if and only if its confluence matrix is of the pairwise diagonal form (2.9).

EXAMPLE 2.1 *(Filtered-X Algorithm with Leakage)* Assume that the adaptive system with harmonic regressor (2.1)-(2.5) is specified as the Filtered-X algorithm [1] with an added leakage term (cf., Ioannou and Kokotovic [11]),

$$\dot{w} = -\sigma w + \mu \tilde{x} e \tag{2.13}$$

$$\tilde{x} = F(p)x \tag{2.14}$$

for some regressor filter $F(s)$, and some value of the leakage parameter $\sigma \geq 0$.

Then, if the XO condition of Theorem 2.1 is satisfied, the LTI expression (2.10) for $\overline{H}$ can be calculated by using the choice $\Gamma(s) = \frac{1}{s+\sigma}$ in Theorem 2.1, to give,

$$\overline{H}(s) = \mu \sum_{i=1}^{m} {d_i}^2 \cdot \frac{F_R(i)(s+\sigma) + F_I(i)\omega_i}{s^2 + 2\sigma s + (\omega_i^2 + \sigma^2)} \tag{2.15}$$

■

Expression (2.15) allows the Filtered-X algorithm to be analyzed in a variety of adaptive applications using standard LTI methods (e.g., Bode plots, Nyquist analysis, root locus, etc.) [3].

REMARK 2.3 In the special case of $\mathcal{X} = I$ one has a paired "sin/cos" regressor $x = c(t)$ which has been studied by many researchers. Specifically, Glover [2] gave a rigorous proof of its LTI properties for $F(s) = 1, \Gamma(s) = 1/s, \sigma = 0$ (in discrete-time), which was later extended by Morgan and Sanford [12] to include $F(s) \neq 1$, and recently by Collins [13] to include a general adaptation law $\Gamma(s)$. Analysis from a control perspective can be found in Sievers and von Flotow [14], Morgan [6], Collins [13], Spanos and Rahman [15], Bodson, Sacks and Khosla [16], and Messner and Bodson [17]. Presently the discrete-time version of this sin/cos result can be found in the book by Widrow and Stearns (cf., [1], page 318).

Unfortunately, the special case $\mathcal{X} = I$ is restrictive in the sense that it requires that the disturbance frequencies are *known* beforehand. In contrast, Theorem 2.1 ensures that expression (2.15) is valid for any $\mathcal{X}$ such that the XO condition holds $\mathcal{X}^T\mathcal{X} = D^2$. For example, the XO condition holds for adaptive systems with infinitely long tap-delay line regressors, (this is equivalent to Glover's original result [2]). A key advantage of XO condition is that it allows treatment of tap-delay lines (in particular), and other types of regressor constructions which are applicable to adaptive noise cancelling of unknown disturbance frequencies. ■

The following result taken from [3][18] shows that in the general case where the XO condition is not satisfied, the mapping $\mathcal{H}$ can always be decomposed into a *parallel connection* of an LTI subsystem and an LTV perturbation.

THEOREM 2.2 (LTI/LTV Decomposition) *Consider the adaptive system (2.1)-(2.3) with harmonic regressor (2.4)(2.5). Then,*

(i) *In general the mapping $\mathcal{H}$ from e to $\hat{y}$ can be expressed as the parallel connection of an*

LTI block $\overline{H}(s)$, and an LTV perturbation block $\tilde{\Delta}$,

$$\mathcal{H}: \quad \hat{y} = \overline{H}(p)e + \tilde{\Delta}[e] \tag{2.16}$$

where,

$$\overline{H}(s) \triangleq \mu \sum_{i=1}^{m} d_i{}^2 \cdot H_i(s) \tag{2.17}$$

$$\tilde{\Delta}[e] \triangleq \mu c(t)^T \Delta \Gamma(p)\Big[\mathcal{F}c(t)e\Big] \tag{2.18}$$

$$\Delta \triangleq \mathcal{X}^T\mathcal{X} - D^2 \tag{2.19}$$

$$\mathcal{F} \triangleq blockdiag\{\mathcal{F}_i\} \in R^{2m\times 2m} \tag{2.20}$$

$$\mathcal{F}_i \triangleq \begin{bmatrix} F_R(i) & F_I(i) \\ -F_I(i) & F_R(i) \end{bmatrix} \in R^{2\times 2}; \qquad for\ i = 1, ..., m \tag{2.21}$$

$$F_R(i) \triangleq \mathbf{Re}(F(j\omega_i)); \quad F_I(i) \triangleq \mathbf{Im}(F(j\omega_i)) \tag{2.22}$$

and where $H_i(s)$ is as defined in (2.11) of Theorem 2.1, and D^2 is chosen (non-uniquely) as any matrix of the 2×2 block-diagonal form (2.9).

(ii) *If the adaptation law $\Gamma(s)$ is stable with infinity norm $||\Gamma(s)||_\infty$, then the gain of the LTV perturbation can be bounded from above as,*

$$||\tilde{\Delta}||_{2i} \leq \mu m \overline{\sigma}(\Delta) ||\Gamma(s)||_\infty \max_i |F(\omega_i)| \tag{2.23}$$

where $||\cdot||_{2i}$ denotes the induced $\mathcal{L}_2$-norm of the indicated operator. ■

3 TAP DELAY-LINE (TDL) BASIS

3.1 Single Tone Case

In 1977, Glover [2] made the interesting and important discovery that an LTI adaptive system arises if its regressor x is constructed by filtering a sinusoid through a very long tap delay line (TDL). Glover's result can be understood in the present context by showing that such a regressor satisfies the XO condition of Theorem 2.1. Specifically, let the regressor $x(t) = [x_1(t), ..., x_N(t)]^T \in R^N$ be defined by filtering a *single* frequency $\omega_1 > 0$,

$$\xi(t) = a_{11}\sin(\omega_1 t) + a_{12}\cos(\omega_1 t) \tag{3.1}$$

through a TDL with N taps and tap delay T, i.e.,

$$x_\ell(t) = e^{-(\ell-1)pT}\xi(t), \quad \ell = 1, ..., N \tag{3.2}$$

where the term $e^{-(\ell-1)pT}$ in the differential operator p represents a delay of $(\ell - 1)T$ time units. If $x(t)$ is written in the form $x = \mathcal{X}c(t)$, then it can be shown (i.e., set $i = 1$ and $m = 1$ in Theorem A.1 of Appendix A), that the confluence matrix is given by,

$$\mathcal{X}^T\mathcal{X} = \frac{N}{2}\alpha_1^2 \cdot I_{2\times 2} + \frac{1}{2}\mathcal{B}_N(\omega_1 T)A_1^T R_{11} A_1 \tag{3.3}$$

where,

$$\alpha_1^2 = a_{11}^2 + a_{12}^2 \tag{3.4}$$

$$A_1 = \begin{bmatrix} a_{12} & -a_{11} \\ a_{11} & a_{12} \end{bmatrix}; \quad R_{11} = \begin{bmatrix} -\cos(N-1)\omega_1 T & \sin(N-1)\omega_1 T \\ \sin(N-1)\omega_1 T & \cos(N-1)\omega_1 T \end{bmatrix} \tag{3.5}$$

$$\mathcal{B}_N(\nu) \triangleq \frac{\sin N\nu}{\sin \nu} \tag{3.6}$$

The first term of (3.3) (a pairwise diagonal matrix), increases as N, while the second term remains bounded. Normalizing the adaptation gain to $\mu = \overline{\mu}/N$ for some $\overline{\mu} > 0$ (to prevent unbounded feedforward gain), the confluence matrix becomes pairwise diagonal in the limit as $N \to \infty$, i.e.,

$$\lim_{N\to\infty} \frac{1}{N}\mathcal{X}^T\mathcal{X} = \frac{1}{2}\alpha_1^2 \cdot I_{2\times 2} \tag{3.7}$$

Hence, the XO condition is satisfied, and according to Theorem 2.1 the system admits an exact LTI representation given by,

$$\mathcal{H}: \quad \hat{y} = \frac{\overline{\mu}}{2}{\alpha_1}^2 \cdot H_1(p)e \tag{3.8}$$

where $H_1(s)$ is given by (2.11)(2.12), and where $\alpha_1^2 = a_{11}^2 + a_{12}^2$. Glover's original result [2] is recovered by specializing (3.8) to $F(s) = 1$ (no regressor filter) and the gradient algorithm $\Gamma(s) = 1/s$.

REMARK 3.1 The result of Elliott [20] on *synchronous sampling* can also be understood in the context of the XO condition. Specifically, if the disturbance frequency ω_1 is known beforehand, the choices of T and N can be *synchronized* in the sense that $N\omega_1 T = k\pi$ for any choice of k such that $0 < k < N$. This ensures that,

$$\mathcal{B}_N(\omega_1 T) = \frac{\sin N\omega_1 T}{\sin \omega_1 T} = 0 \tag{3.9}$$

Using (3.9) in (3.3) shows that,

$$\mathcal{X}^T \mathcal{X} = \frac{N}{2}\alpha_1^2 \cdot I_{2\times 2} \tag{3.10}$$

Hence, the XO condition is satisfied, and according to Theorem 2.1 the system admits an exact LTI representation with only a finite number of taps. ∎

3.2 Multi-Tone Case

Rigorously speaking, Glover's LTI analysis of TDL regressors is only applicable in the limiting case when $N = \infty$, i.e., it ignores the contribution of the LTV subsystem for finite N. A more complete solution can be found by putting Glover's results into a modern robust control setting. This will be done in the present section by applying the LTI/LTV decomposition of Theorem 2.2 to the TDL regressor case. First, a definition will be useful.

DEFINITION 3.1 *Given time delay T and spacing parameter $0 < \underline{\nu} < \pi/2$, a Bounded Tone Set $\Omega(m, T, \underline{\nu})$ is defined as any set of m frequencies $\{\omega_i\}_{i=1}^m$, such that,*

$$\Omega(m,T,\underline{\nu}) \triangleq \left\{ \begin{array}{ll} \{\omega_i\}_{i=1}^m : & 0 < \underline{\nu} < \pi/2; \\ 0 < \underline{\nu} < \omega_i T \le \pi - \underline{\nu} & \text{for all } i = 1, \ldots, m; \\ |\omega_i - \omega_j| T \ge 2\underline{\nu} & \text{for all } i \ne j \end{array} \right\} \tag{3.11}$$

∎

Simply stated, a Bounded Tone Set is a set of frequencies $\{\omega_i\}_{i=1}^m$ which are bounded away from 0, π/T and each other. The definition is not very restrictive since any signal comprised of a finite number of distinct sinusoids lies in a Bounded Tone Set when T is chosen sufficiently small (i.e., to ensure Nyquist sampling of its highest component). The *minimum spacing* parameter $\underline{\nu}$ will play a central role in subsequent discussion.

The main result of the paper follows [3][4].

THEOREM 3.1 (Tap Delay-Line Basis) *Consider the adaptive system (2.1)-(2.3) with harmonic regressor (2.4)(2.5), and input/output mapping $\mathcal{H}$ in (2.6). Let the components of the regressor $x = [x_1, ..., x_N]^T \in R^N$ be defined by filtering a signal $\xi(t) \in R^1$ through a tap delay line with N taps and tap delay T, i.e.,*

$$x_\ell = e^{-(\ell-1)pT}\xi, \quad \ell = 1, ..., N \tag{3.12}$$

where the measured signal ξ is given by the following sum of m sinusoids,

$$\xi(t) = \sum_{i=1}^{m} \alpha_i \sin(\omega_i t + \phi_i); \quad \alpha_i > 0 \tag{3.13}$$

and frequencies $\{\omega_i\}_{i=1}^{m}$ lie in a bounded tone set $\Omega(m, T, \underline{\nu})$.

Then,

(i) *The regressor $x(t)$ can be written in harmonic form (2.4)(2.5) where the matrix $\mathcal{X} \in R^{N\times 2m}$ satisfies,*

$$\mathcal{X}^T\mathcal{X} = D^2 + \Delta \tag{3.14}$$

$$D^2 = \frac{N}{2}\begin{bmatrix} \alpha_1^2 \cdot I_{2\times 2} & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \alpha_m^2 \cdot I_{2\times 2} \end{bmatrix} \in R^{2m\times 2m} \tag{3.15}$$

and the matrix perturbation $\Delta \triangleq \mathcal{X}^T\mathcal{X} - D^2$ is norm-bounded as,

$$\overline{\sigma}(\Delta) \le \frac{m\pi\alpha_{max}^2}{2\underline{\nu}}; \quad \alpha_{max} \triangleq \max_i\{\alpha_i\} \tag{3.16}$$

(ii) *(LTI/LTV Decomposition)*

The mapping $\mathcal{H}$ from $\hat{y}$ to e can be uniquely decomposed into the parallel connection of an LTI block $\overline{H}(s)$, and an LTV perturbation block $\tilde{\Delta}$,

$$\mathcal{H}: \quad \hat{y} = \overline{H}(p)e + \tilde{\Delta}[e] \tag{3.17}$$

where,

$$\overline{H}(s) \triangleq \mu\frac{N}{2}\sum_{i=1}^{m}\alpha_i^2 \cdot H_i(s) \tag{3.18}$$

$$\tilde{\Delta}[e] \triangleq \mu c(t)^T \Delta\Gamma(p)\Big[\mathcal{F}c(t)e\Big] \tag{3.19}$$

where the perturbation matrix Δ is defined in (3.14) with norm bound (3.16), and $H_i(s)$ is given by (2.11) of Theorem 2.1.

Furthermore, if the adaptation law $\Gamma(s)$ is stable with infinity norm $||\Gamma(s)||_\infty$, then the gain of the LTV perturbation can be bounded as,

$$||\tilde{\Delta}||_{2i} \leq \frac{\mu m^2 \pi}{2\underline{\nu}} \cdot \left(\alpha_{max}^2 ||\Gamma(s)||_\infty \max_i |F(\omega_i)|\right) \tag{3.20}$$

where $||\cdot||_{2i}$ indicates the induced $\mathcal{L}_2$-norm.

(iii) *(Normalized Adaptation Gain)*

By normalizing the adaptive gain to $\mu = \overline{\mu}/N$, the operators in (3.18)(3.19) of **(ii)** *become,*

$$\overline{H}(s) = \frac{\overline{\mu}}{2}\sum_{i=1}^{m}\alpha_i^2 \cdot H_i(s) \tag{3.21}$$

$$\tilde{\Delta}[e] \triangleq \frac{\overline{\mu}}{N}c(t)^T \Delta\Gamma(p)\Big[\mathcal{F}c(t)e\Big] \tag{3.22}$$

and the upper bound on the gain of the LTV perturbation in (3.20) becomes,

$$||\tilde{\Delta}||_{2i} \leq \frac{\overline{\mu} m^2 \pi}{2N\underline{\nu}} \cdot \left(\alpha_{max}^2 ||\Gamma(s)||_\infty \max_i |F(\omega_i)|\right) \tag{3.23}$$

where $||\Gamma(s)||_\infty$ is assumed to exist.

(iv) *(Asymptotic Properties)*

If the adaptation law $\Gamma(s)$ is stable (with bounded infinity norm $||\Gamma(s)||_\infty$), and the adaptation gain is normalized as $\mu = \overline{\mu}/N$ for $\overline{\mu} > 0$ constant, then as $N \to \infty$ the mapping $\mathcal{H}$ becomes LTI with asymptotic transfer function,

$$\overline{H}(s) = \frac{\overline{\mu}}{2}\sum_{i=1}^{m} \alpha_i^2 \cdot H_i(s) \tag{3.24}$$

PROOF:

Proof of **(i)**: From Theorem A.1 the confluence matrix is given by $\mathcal{X}^T\mathcal{X} = \mathcal{M}$ where $\mathcal{M}$ is given by (A.45)-(A.48). Hence, $\Delta = \mathcal{M} - D^2 \in R^{2m\times 2m}$ has the symmetric block 2-by-2 structure (A.70) used in Lemma A.5. Applying the result (A.71) of Lemma A.5 gives,

$$\begin{aligned} \overline{\sigma}(\Delta) \;&\leq\; m \cdot \max_{i,j} \overline{\sigma}(\Delta_{ij}) \leq m \cdot \max_{i,j} \frac{\pi \alpha_i \alpha_j}{2\underline{\nu}} && (3.25) \\ &\leq\; \frac{m\pi}{2\underline{\nu}} \max_i \{\alpha_i^2\} \triangleq \frac{m\pi}{2\underline{\nu}} \alpha_{max}^2 && (3.26) \end{aligned}$$

where use has been made of property **P.5** of Lemma A.4 in equation (3.25).

Proof of **(ii)**: Results follow by applying the LTI/LTV Decomposition of Theorem 2.2 noting from (3.15) that in the present case ${d_i}^2 = \frac{N}{2}\alpha_i^2$.

Proof of **(iii)**: Simply substitute $\mu = \overline{\mu}/N$ into the LTI and LTV blocks, where $\overline{\mu} > 0$ is a constant.

Proof of **(iv)**: It is seen that the normalized LTI transfer function $\overline{H}(s)$ in (3.21) remains unaffected as N increases while the normalized LTV perturbation in (3.22) goes to zero as N increases. Hence, as $N \to \infty$ the mapping $\mathcal{H}$ becomes LTI with asymptotic transfer function given in (3.24), as desired. ∎

For convenience, the results of Theorem 3.1 are summarized in Figure 3.1. Specifically, Figure 3.1 Part a. shows the harmonic adaptive system with TDL basis and normalized adaptation gain $\mu = \overline{\mu}/N$; Part b. shows the equivalent decomposition into an LTI block and a norm bounded LTV perturbation block. Note that the time-varying perturbation block goes to zero asymptotically as N becomes large.

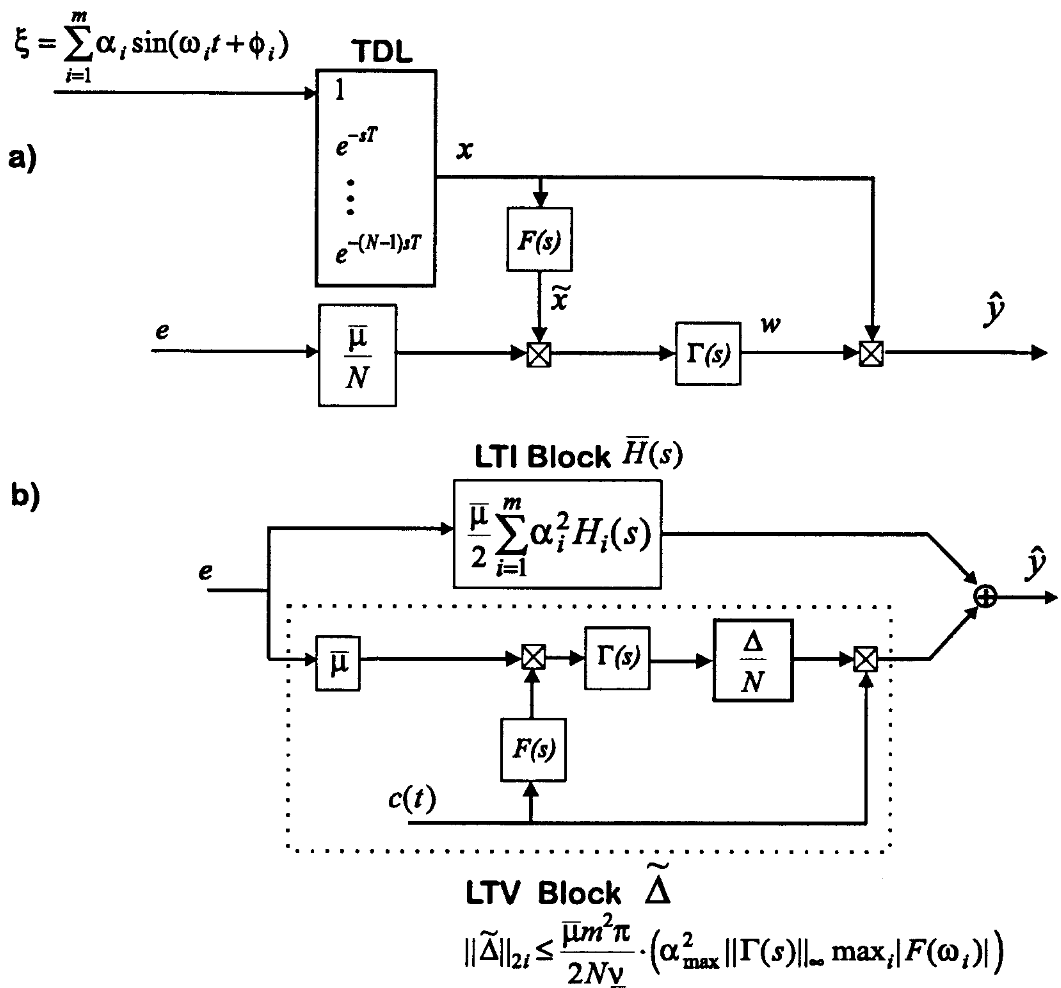

Figure 3.1: LTI/LTV decomposition of $\mathcal{H}$ for harmonic adaptive system with TDL basis

REMARK 3.2 The asymptotic result **(iv)** of Theorem 3.1 follows essentially from the special form of the confluence matrix $\mathcal{X}^T\mathcal{X}$ in (3.14) which arises in the TDL case. Specifically, in relation (3.14), the matrix D^2 given by (3.15) (and hence the associated LTI block) *grows linearly* with the number of taps N, while the perturbation matrix Δ (and hence the associated LTV block) *remains bounded* as N increases.

Hence, when the LTI and LTV paths are normalized by $1/N$ through choice of adaptation gain $\mu = \overline{\mu}/N$ (as shown in Figure 3.1) and the limit is taken as N becomes large, the LTI part remains constant while the norm bound on the LTV part decreases as $1/N$. This indicates that the LTV part can be made arbitrarily small by choosing N sufficiently large, while the LTI part remains unaffected. ∎

REMARK 3.3 Interestingly, the bound on the LTV perturbation $\tilde{\Delta}$ in (3.23) depends on the boundedness of the tone-set through the minimum spacing parameter $\underline{\nu} > 0$ defined in Definition 3.1. Specifically, a smaller $\underline{\nu}$ requires a larger N to justify the asymptotic approximation to the same degree. It is worth noting that these analytical results are consistent with the heuristic discussions of tone spacing found Glover's original paper [2] (cf., Section IV, pp. 488). Also interesting is the appearance of m^2 in the numerator of the norm bound (3.23) which indicates that if the number of tones m in the regressor is increased, one must increase N as the *square* of m to justify the asymptotic approximation to the same degree. ∎

EXAMPLE 3.1 *(Filtered-X Algorithm with Leakage)* Assume that the adaptive system in Theorem 3.1 is specified as the Filtered-X algorithm with an added leakage term (2.13)(2.14). Let the regressor x be defined as in Theorem 3.1 by filtering the sinusoidal signal $\xi(t)$ in (3.13) through a tap delay line with N taps and tap delay T, and where the frequencies $\{\omega_i\}_{i=1}^m$ in $\xi(t)$ lie in a bounded tone set $\Omega(m, T, \underline{\nu})$. Then using a normalized adaptive gain $\mu = \overline{\mu}/N$, the LTI expression (3.21) for $\overline{H}$ can be calculated as,

$$\overline{H}(s) = \frac{\overline{\mu}}{2} \sum_{i=1}^{m} {\alpha_i}^2 \cdot \frac{F_R(i)(s+\sigma) + F_I(i)\omega_i}{s^2 + 2\sigma s + (\omega_i^2 + \sigma^2)} \tag{3.27}$$

The norm bound (3.23) on the LTV perturbation is computed as,

$$||\tilde{\Delta}||_{2i} \leq \frac{\overline{\mu} m^2 \pi}{2N\underline{\nu}\sigma} \cdot \left(\alpha_{max}^2 \max_i |F(\omega_i)| \right) \tag{3.28}$$

■

REMARK 3.4 It is emphasized that using the LTI transfer function (3.27) and the norm bounded perturbation (3.28), the Filtered-X algorithm with a *finite-length TDL* regressor can be analyzed in adaptive applications using a variety of standard modern robust control methods (e.g., small gain theorem, μ-Synthesis, etc.). This includes the important practical case where the plant $P(s)$ blocking the cancellation path is not completely known, i.e., the regressor filter is chosen as $F(s) = \hat{P}(s)$ where $P(s) = \hat{P}(s)(1 + \Delta_M)$ and Δ_M is a specified multiplicative uncertainty (for example). ■

REMARK 3.5 The bound (3.28) on the LTV perturbation is finite only if $\sigma \neq 0$. Accordingly, "leakage" is required in the adaptive law to ensure its asymptotic convergence to an LTI system as the number of taps is increased. Interestingly, the need for leakage in this context has not been previously considered by Glover and others. ■

4 CONCLUSIONS

A 1977 result due to Glover indicates that an adaptive system with a sinusoidal tap-delay-line regressor becomes LTI in the limit as the number of taps is increased to infinity. This result forms the basis for understanding and designing many present-day adaptive algorithms which are able to cancel sinusoidal disturbances of *unknown* frequency [1].

Unfortunately, Glover's result is rigorously only true in the limit as the number of taps is increased to infinity. The present paper (cf., Theorem 3.1) extends these earlier results by putting the problem into a modern robust control setting. Specifically, the adaptive system is shown to be a parallel connection of LTI and LTV blocks. Since one is mainly interested in the LTI properties, the LTV part is treated as a perturbation. The main result of this paper computes an explicit norm-bound (3.23) on the LTV perturbation. The norm-bound is seen to be proportional to $m^2/(N\underline{\nu})$ which clearly indicates its size as a function of the number of taps N, the minimum tone spacing parameter $\underline{\nu}$, and the number of tones m. The norm bound goes to zero asymptotically as N goes to infinity thus recovering Glover's result. However, in the more general case where N is finite, the availability of this norm-bound opens up new opportunities for analyzing adaptive systems using modern robust control methods, applicable to LTI systems subject to norm-bounded perturbations.

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A APPENDIX: Properties of TDL Regressors

The purpose of Appendix A is to provide the detailed structure of the confluence matrix $\mathcal{X}^T\mathcal{X}$ for a tap delay-line basis (i.e., in Theorem A.1), and additional supporting results which are needed to prove Theorem 3.1.

The following definitions will be used throughout Appendix A:

$$C_i \triangleq \begin{bmatrix} 1 \\ \cos\omega_i T \\ \vdots \\ \cos(N-1)\omega_i T \end{bmatrix} \in R^N; \qquad S_i \triangleq \begin{bmatrix} 0 \\ \sin\omega_i T \\ \vdots \\ \sin(N-1)\omega_i T \end{bmatrix} \in R^N \tag{A.1}$$

$$\underline{c}_{ij} \triangleq \cos((N-1)(\omega_i-\omega_j)T/2) \tag{A.2}$$

$$\underline{s}_{ij} \triangleq \sin((N-1)(\omega_i-\omega_j)T/2) \tag{A.3}$$

$$c_{ij} \triangleq \cos((N-1)(\omega_i+\omega_j)T/2) \tag{A.4}$$

$$s_{ij} \triangleq \sin((N-1)(\omega_i+\omega_j)T/2) \tag{A.5}$$

$$\underline{R}_{ij} \triangleq \begin{bmatrix} \underline{c}_{ij} & \underline{s}_{ij} \\ -\underline{s}_{ij} & \underline{c}_{ij} \end{bmatrix} \tag{A.6}$$

$$R_{ij} \triangleq \begin{bmatrix} -c_{ij} & s_{ij} \\ s_{ij} & c_{ij} \end{bmatrix} \tag{A.7}$$

$$\xi(t) \triangleq \sum_{i=1}^{m} \alpha_i \sin(\omega_i t + \phi_i) = \sum_{i=1}^{m} a_{i1} \sin(\omega_i t) + a_{i2} \cos(\omega_i t) \tag{A.8}$$

$$A_i \triangleq \begin{bmatrix} a_{i2} & -a_{i1} \\ a_{i1} & a_{i2} \end{bmatrix} \tag{A.9}$$

$$\mathcal{B}_N(\nu) \triangleq \frac{\sin N\nu}{\sin \nu} \tag{A.10}$$

LEMMA A.1 *Let $\mathcal{B}_N(\nu)$ be defined by (A.10). Then, on the interval $0 \le \nu \le \pi$, the following inequality holds,*

$$|\mathcal{B}_N(\nu)| \le \frac{\pi}{2\tau(\nu)}; \quad \text{for } 0 \le \nu \le \pi \tag{A.11}$$

where,

$$\tau(\nu) \triangleq \min(\nu, \pi - \nu) \tag{A.12}$$

PROOF: A sinusoid $\sin\nu$ can be bounded below on the interval $0 \le \nu \le \pi$ by piecewise linear segments as follows,

$$|\sin\nu| \ge \frac{1}{\pi} \min(2\nu, 2(\pi - \nu)); \quad \text{for } 0 \le \nu \le \pi \tag{A.13}$$

Hence,

$$\left|\frac{\sin N\nu}{\sin \nu}\right| \le \frac{\pi |\sin N\nu|}{\min(2\nu, 2(\pi - \nu))} \le \frac{\pi}{2 \cdot \tau(\nu)} \tag{A.14}$$

■

LEMMA A.2 *Let $\mathcal{B}_N(\nu)$ be defined by (A.10). Then for frequencies $\{\omega_i\}_{i=1}^{m}$ in a bounded tone set $\Omega(m, T, \underline{\nu})$, the following inequalities hold,*

$$|\mathcal{B}_N(\omega_i T)| \qquad \le \frac{\pi}{2\underline{\nu}} \tag{A.15}$$

$$|\mathcal{B}_N((\omega_i - \omega_j)T/2)| \ \leq \frac{\pi}{2\underline{\nu}} \tag{A.16}$$

$$|\mathcal{B}_N((\omega_i + \omega_j)T/2)| \ \leq \frac{\pi}{2\underline{\nu}} \tag{A.17}$$

PROOF: By properties of the Bounded Tone Set (3.11), the following inequalities can be shown to hold for any $\omega_i, \omega_j \in \Omega(m, T, \underline{\nu})$,

$$2\underline{\nu} \leq |\omega_i - \omega_j|T \leq \pi - 2\underline{\nu} \tag{A.18}$$

$$2\underline{\nu} \leq |\omega_i + \omega_j|T \leq 2\pi - 2\underline{\nu} \tag{A.19}$$

Proof of (A.15):

$$\mathcal{B}_N(\omega_i T) \leq \frac{\pi}{2\tau(\omega_i T)} = \frac{\pi}{2\min(\omega_i T, \pi - \omega_i T)} \leq \frac{\pi}{2\underline{\nu}} \tag{A.20}$$

Here, the first inequality in (A.20) follows by (A.11) of Lemma A.1; the second equality follows by definition of τ in (A.12); and the last inequality follows by properties of the bounded tone set $\Omega(m, T, \underline{\nu})$ in (3.11).

Proof of (A.16):

$$\begin{aligned}
\mathcal{B}_N((\omega_i - \omega_j)T/2) \ &\leq \ \frac{\pi}{2\tau((\omega_i - \omega_j)T/2)} && \text{(A.21)}\\
&= \ \frac{\pi}{\min(|\omega_i - \omega_j|T, 2\pi - |\omega_i - \omega_j|T)} && \text{(A.22)}\\
&\leq \ \frac{\pi}{\min(2\underline{\nu}, 2\pi - (\pi - 2\underline{\nu}))} = \frac{\pi}{2\underline{\nu}} && \text{(A.23)}
\end{aligned}$$

Here, equation (A.21) follows by (A.11); (A.22) follows by the definition of $\tau(\cdot)$ in (A.12) and the fact that the function $\mathcal{B}_N(\cdot)$ is even; and (A.23) follows from (A.18).

Proof of (A.17):

$$\begin{aligned}
\mathcal{B}_N((\omega_i + \omega_j)T/2) \ &\leq \ \frac{\pi}{2\tau((\omega_i + \omega_j)T/2)} && \text{(A.24)}\\
&= \ \frac{\pi}{\min(|\omega_i + \omega_j|T, 2\pi - |\omega_i + \omega_j|T)} && \text{(A.25)}\\
&\leq \ \frac{\pi}{\min(2\underline{\nu}, 2\pi - (2\pi - 2\underline{\nu}))} = \frac{\pi}{2\underline{\nu}} && \text{(A.26)}
\end{aligned}$$

Here, equation (A.24) follows by (A.11); (A.25) follows by the definition of $\tau(\cdot)$ in (A.12) and the fact that the function $\mathcal{B}_N(\cdot)$ is even; and (A.26) follows from (A.19). ■

LEMMA A.3 *Let C_i, S_i be as defined in (A.1). Then the following identities hold,*

$$S_i^T S_j = \frac{1}{2}\Big(\underline{c}_{ij}\cdot\mathcal{B}_N\left((\omega_i-\omega_j)T/2\right) - c_{ij}\cdot\mathcal{B}_N\left((\omega_i+\omega_j)T/2\right)\Big) \tag{A.27}$$

$$C_i^T C_j = \frac{1}{2}\Big(\underline{c}_{ij}\cdot\mathcal{B}_N\left((\omega_i-\omega_j)T/2\right) + c_{ij}\cdot\mathcal{B}_N\left((\omega_i+\omega_j)T/2\right)\Big) \tag{A.28}$$

$$C_j^T S_i = S_i^T C_j = \frac{1}{2}\Big(s_{ij}\cdot\mathcal{B}_N\left((\omega_i+\omega_j)T/2\right) + \underline{s}_{ij}\cdot\mathcal{B}_N\left((\omega_i-\omega_j)T/2\right)\Big) \tag{A.29}$$

$$C_i^T S_j = S_j^T C_i = \frac{1}{2}\Big(s_{ij}\cdot\mathcal{B}_N\left((\omega_i+\omega_j)T/2\right) - \underline{s}_{ij}\cdot\mathcal{B}_N\left((\omega_i-\omega_j)T/2\right)\Big) \tag{A.30}$$

where $\underline{c}_{ij}, \underline{s}_{ij}, s_{ij}, c_{ij}, \mathcal{B}_N$ are as defined in (A.2)-(A.5), and (A.10).

PROOF: Use will be made of the following identity,

$$\sum_{\ell=1}^{N} e^{j(\ell-1)\nu} = \frac{1-e^{jN\nu}}{1-e^{j\nu}} \tag{A.31}$$

$$= e^{j(N-1)\nu/2}\cdot\frac{\sin(N\nu/2)}{\sin(\nu/2)} \triangleq e^{j(N-1)\nu/2}\cdot\mathcal{B}_N(\nu/2) \tag{A.32}$$

Proof of (A.27):

$$S_i^T S_j = \sum_{\ell=1}^{N}\sin\Big(\omega_i(\ell-1)T\Big)\sin\Big(\omega_j(\ell-1)T\Big) \tag{A.33}$$

$$= \frac{1}{2}\sum_{\ell=1}^{N}\cos\Big((\omega_i-\omega_j)(\ell-1)T\Big) - \cos\Big((\omega_i+\omega_j)(\ell-1)T\Big) \tag{A.34}$$

$$= \frac{1}{2}\mathbf{Re}\left\{\sum_{\ell=1}^{N} e^{j(\omega_i-\omega_j)(\ell-1)T} - e^{j(\omega_i+\omega_j)(\ell-1)T}\right\} \tag{A.35}$$

$$= \frac{1}{2}\mathbf{Re}\Big\{e^{j(N-1)(\omega_i-\omega_j)T/2}\mathcal{B}_N((\omega_i-\omega_j)T/2)$$

$$- \quad e^{j(N-1)(\omega_i+\omega_j)T}\mathcal{B}_N((\omega_i+\omega_j)T/2)\Big\} \tag{A.36}$$

$$= \frac{1}{2}\Big(\underline{c}_{ij}\mathcal{B}_N((\omega_i-\omega_j)T/2) - c_{ij}\mathcal{B}_N((\omega_i+\omega_j)T/2)\Big) \tag{A.37}$$

Here, $\mathbf{Re}(\cdot)$ denotes the real part of the indicated expression. Equations (A.33)-(A.35) follow by standard trig formulas; equation (A.36) follows by using identity (A.32); and (A.37) follows by the definition of s_{ij} and $\underline{s}_{ij}$ in (A.2)-(A.5).

Proof of (A.28)(A.29): These proofs follow by a sequence of rearrangements similar to those performed in (A.33)-(A.37), but starting with the trigonometric expressions,

$$C_i^T C_j = \sum_{\ell=1}^{N} \cos\Big(\omega_i(\ell-1)T\Big) \cos\Big(\omega_j(\ell-1)T\Big) \tag{A.38}$$

$$= \frac{1}{2}\mathbf{Re}\Big\{\sum_{\ell=1}^{N} e^{j(\omega_i-\omega_j)(\ell-1)T} + e^{j(\omega_i+\omega_j)(\ell-1)T}\Big\} \tag{A.39}$$

$$C_j^T S_i = S_i^T C_j = \sum_{\ell=1}^{N} \sin\Big(\omega_i(\ell-1)T\Big) \cos\Big(\omega_j(\ell-1)T\Big) \tag{A.40}$$

$$= \frac{1}{2}\mathbf{Im}\Big\{\sum_{\ell=1}^{N} e^{j(\omega_i+\omega_j)(\ell-1)T} + e^{j(\omega_i-\omega_j)(\ell-1)T}\Big\} \tag{A.41}$$

Proof of (A.30): This relation follows by reversing the roles of i and j in the proof of (A.29), making use of the antisymmetric property of $\sin(-\theta) = -\sin(\theta)$ and symmetric property of $\mathcal{B}_N(-\nu) = \mathcal{B}_N(\nu)$. ∎

THEOREM A.1 (Confluence Matrix for a TDL) *Let the components of the regressor $x = [x_1, ..., x_N]^T \in R^N$ be defined by filtering a signal $\xi(t) \in R^1$ through a tap delay line with N taps and tap delay T, i.e.,*

$$x_\ell = e^{-(\ell-1)sT}\xi, \quad \ell = 1, ..., N \tag{A.42}$$

where the measured signal ξ is given by the following sum of m distinct sinusoids,

$$\xi(t) = \sum_{i=1}^{m} \alpha_i \sin(\omega_i t + \phi_i) = \sum_{i=1}^{m} a_{i1}\sin(\omega_i t) + a_{i2}\cos(\omega_i t) \tag{A.43}$$

Then, the regressor $x(t)$ is of the harmonic form,

$$x = \mathcal{X}c(t) \tag{A.44}$$

where $\mathcal{X} \in R^{N \times 2m}$, $c(t)$ is defined in (2.5), and its confluence matrix is given by,

$$\mathcal{X}^T \mathcal{X} = \begin{bmatrix} \mathcal{M}_{11} & \dots & \mathcal{M}_{1m} \\ \vdots & \ddots & \vdots \\ \mathcal{M}_{m1} & \dots & \mathcal{M}_{mm} \end{bmatrix} \in R^{2m \times 2m} \tag{A.45}$$

$$\mathcal{M}_{ij} \in R^{2 \times 2}; \quad i = 1, ..., m; \; j = 1, ..., m \tag{A.46}$$

$$\mathcal{M}_{ij} = \frac{1}{2} A_i^T \left(\mathcal{B}_N \left((\omega_i - \omega_j) T/2 \right) \underline{R}_{ij} + \mathcal{B}_N \left((\omega_i + \omega_j) T/2 \right) R_{ij} \right) A_j \tag{A.47}$$

$$\mathcal{M}_{ii} = \frac{N}{2} \alpha_i^2 \cdot I_{2 \times 2} + \frac{1}{2} \mathcal{B}_N(\omega_i T) A_i^T R_{ii} A_i \tag{A.48}$$

where definitions (A.1)-(A.10) have been used.

PROOF: Using standard trigonometric identities, the ℓth element x_ℓ of the delayed regressor (A.42) can be expanded as,

$$x_\ell(t) = e^{-(\ell-1)sT} \xi = \sum_{i=1}^{m} a_{i1} \sin\Big(\omega_i(t - (\ell-1)T)\Big) + a_{i2} \cos\Big(\omega_i(t - (\ell-1)T)\Big) \tag{A.49}$$

$$\begin{aligned} &= \sum_{i=1}^{m} a_{i1} \Big(\cos(\omega_i(\ell-1)T) \sin \omega_i t - \sin(\omega_i(\ell-1)T) \cos \omega_i t \Big) \\ &+ \; a_{i2} \Big(\sin(\omega_i(\ell-1)T) \sin \omega_i t + \cos(\omega_i(\ell-1)T) \cos \omega_i t \Big) \end{aligned} \tag{A.50}$$

Using (A.50), the full regressor $x(t)$ can be decomposed in terms of the vectors S_i and C_i in (A.1) as follows,

$$x(t) = \Big[a_{12} S_1 + a_{11} C_1, -a_{11} S_1 + a_{12} C_1, ..., a_{m2} S_m + a_{m1} C_m, -a_{m1} S_m + a_{m2} C_m \Big] c(t) \tag{A.51}$$

Equivalently, (in matrix notation),

$$x(t) = Q \mathcal{A} c(t) \tag{A.52}$$

where,

$$Q \triangleq [S_1, C_1, ..., S_m, C_m] \in R^{N \times 2m} \tag{A.53}$$

$$\mathcal{A} \triangleq \begin{bmatrix} A_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & A_m \end{bmatrix} \in R^{2m \times 2m} \tag{A.54}$$

and $A_i \in R^{2\times 2}$ is defined as in (A.9). Hence, by inspection of (A.44) and (A.52), the matrix $\mathcal{X}$ is given as,

$$\mathcal{X} = Q\mathcal{A} \tag{A.55}$$

Squaring up $\mathcal{X}$ and using (A.55) gives the confluence matrix,

$$\mathcal{X}^T\mathcal{X} = \mathcal{A}^T Q^T Q \mathcal{A} \triangleq \mathcal{M} \tag{A.56}$$

From the structure of (A.53)-(A.55), the components blocks of $\mathcal{M}$ can be computed as,

$$\mathcal{M}_{ij} = A_i^T \begin{bmatrix} S_i^T S_j & S_i^T C_j \\ C_i^T S_j & C_i^T C_j \end{bmatrix} A_j \tag{A.57}$$

Result (A.47) follows by substituting the identities (A.27)-(A.30) of Lemma A.3 into (A.57) and simplifying using expressions (A.1)-(A.10). Result (A.48) follows by setting $i = j$ in result (A.47) and simplifying by using the relation $\mathcal{B}_N(0) = N$. ∎

LEMMA A.4 *Define,*

$$\Delta \triangleq \mathcal{M} - D^2 = \{\Delta_{ij}\} \in R^{2m\times 2m} \tag{A.58}$$

$$\Delta_{ii} \triangleq \mathcal{M}_{ii} - \frac{N}{2}\alpha_i^2 \cdot I_{2\times 2} \in R^{2\times 2} \tag{A.59}$$

$$\Delta_{ij} \triangleq \mathcal{M}_{ij} \in R^{2\times 2} \tag{A.60}$$

where D^2 is defined by in (3.15) of Theorem 3.1, and the matrix $\mathcal{M}$ and its submatrices $\mathcal{M}_{ij}, \mathcal{M}_{ii}$ are defined by (A.45)(A.48) of Theorem A.1.

Let the quantities $\alpha_i, A_i, \underline{R}_{ij}, R_{ij}$ be defined by (A.6)-(A.9), and assume that all frequencies $\{\omega_i\}_{i=1}^m$ are drawn from a bounded tone set i.e., $\Omega(m, T, \underline{\nu})$ defined in Definition 3.1. Then the following properties hold,

P1. $\overline{\sigma}(A_i) = \alpha_i$

P2. $\overline{\sigma}(\underline{R}_{ij}) = 1$

P3. $\overline{\sigma}(R_{ij}) = 1$

P4. $\overline{\sigma}(\Delta_{ii}) \leq \frac{\pi\alpha_i^2}{2\underline{\nu}}$

P5. $\overline{\sigma}(\Delta_{ij}) \leq \frac{\pi\alpha_i\alpha_j}{2\underline{\nu}}$

PROOF: It follows from the definition of ξ in (A.8), that the variables a_{i1}, a_{i2}, α_i are related as

$$\alpha_i^2 = a_{i1}^2 + a_{i2}^2 \tag{A.61}$$

For the proof, extensive use will be made of the definitions (A.2)-(A.10). Continuing,

Proof of P1:

$$\overline{\sigma}(A_i) = \lambda_{max}^{\frac{1}{2}}(A_i^T A_i) = \lambda_{max}^{\frac{1}{2}}\Big[(a_{i1}^2 + a_{i2}^2)\cdot I\Big] = \alpha_i \tag{A.62}$$

Proof of P2:

$$\overline{\sigma}(\underline{R}_{ij}) = \lambda_{max}^{\frac{1}{2}}(\underline{R}_{ij}^T \underline{R}_{ij}) = \lambda_{max}^{\frac{1}{2}}\Big[(\underline{c}_{ij}^2 + \underline{s}_{ij}^2)\cdot I\Big] = 1 \tag{A.63}$$

Proof of P3:

$$\overline{\sigma}(R_{ij}) = \lambda_{max}^{\frac{1}{2}}(R_{ij}^T R_{ij}) = \lambda_{max}^{\frac{1}{2}}\Big[(c_{ij}^2 + s_{ij}^2)\cdot I\Big] = 1 \tag{A.64}$$

Proof of P4:

$$\overline{\sigma}(\Delta_{ii}) = \overline{\sigma}(\mathcal{M}_{ii} - \frac{N}{2}\alpha_i \cdot I_{2\times 2}) = \overline{\sigma}\left(\frac{1}{2}\mathcal{B}_N(\omega_i T) A_i^T R_{ii} A_i\right) \tag{A.65}$$

$$\leq \frac{1}{2}|\mathcal{B}_N(\omega_i T)| \cdot \overline{\sigma}(A_i)^2 \leq \frac{\pi\alpha_i^2}{4\underline{\nu}} \tag{A.66}$$

where (A.65) follows by (A.48) of Theorem A.1; equation (A.66) follows by P3 for $i = j$; and the last inequality follows by Lemma A.2 and property P1 proved above.

Proof of P5:

$$\begin{aligned} \overline{\sigma}(\Delta_{ij}) &= \overline{\sigma}\left(\frac{1}{2}\mathcal{B}_N\left((\omega_i - \omega_j)T/2\right) A_i^T \underline{R}_{ij} A_j + \mathcal{B}_N\left((\omega_i + \omega_j)T/2\right) A_i^T R_{ij} A_j\right) \quad \text{(A.67)} \\ &\leq \frac{1}{2}\overline{\sigma}(A_i)\cdot\overline{\sigma}(A_j)\Big(|\mathcal{B}_N\left((\omega_i - \omega_j)T/2\right)| \cdot \overline{\sigma}(\underline{R}_{ij}) \end{aligned}$$

$$+ \quad |\mathcal{B}_N((\omega_i+\omega_j)T/2)| \cdot \overline{\sigma}(R_{ij})\Big) \tag{A.68}$$

$$= \quad \frac{\pi\alpha_i\alpha_j}{2\underline{\nu}} \tag{A.69}$$

Here, (A.67) follows from (A.47) of Theorem A.1 and (A.60); and (A.69) follows by Lemma A.2 and properties P1, P2, and P3 proved above. ∎

LEMMA A.5 *Let $X = X^T \in R^{2m \times 2m}$ be a symmetric matrix partitioned into 2×2 blocks, i.e.,*

$$X = \begin{bmatrix} X_{11} & \dots & X_{1m} \\ \vdots & \ddots & \vdots \\ X_{m1} & \dots & X_{mm} \end{bmatrix} \in R^{2m \times 2m}; \quad X_{ij} \in R^{2\times 2} \tag{A.70}$$

Then,

$$\overline{\sigma}(X) \leq m \cdot \max_{ij} \overline{\sigma}(X_{ij}) \tag{A.71}$$

PROOF: See Lemma B.5 of [3]. ∎

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